Theoretical Physics is Magic

Smeapancol

Unit 1: Space

Lesson 1- Algebra

Day 5- Polynomials

The next day, Twilight found Luna in the Canterlot gardens. Luna was in repose under a palm frond, apparently enjoying the scents of nature.

"Princess? Are you here?"

"Under here, disciple!"

Twilight found Luna in repose under an enormous palm frond. Little flowers grew all around her. Light filtered in between the leaves of the frond but it was still dim. It was just like a little tent. Time stood still inside.

"Good morning," said Twilight as she crouched under the palm frond. "This is a beautiful spot."

"Yes. This is our secret hideout. Celestia does not know about it," said Luna. "Today we must do a little detour."

"I think a detour would require us to be actually on track in the first place," said Twilight.

"Why, whatever do you mean?"

"I mean, we haven't done any actual physics yet. We've just been wandering aimlessly through arcane mathematics."

"Yesterday we did rotations, and those are the most basic, most important concept in geometry, which is the first principle of physics. You should not complain. If you jump into physics too quickly, you will learn nothing deeply. You must trust your tutor to lead you along the right track."

"Very well, Princess. I trust you," Twilight might have argued a little more, but it was such a beautiful day.

"Very good. Now then." At that moment Luna conjured a notebook of graph paper. Her horn glowed and symbols began appearing on the paper.

$$p(x) = a_0 + a_1\, x + a_2\, x^2 \ldots \tag{5.1}$$

"That's a polynomial," she said

"I know," said Twilight.

"Do you know what it means to factor a polynomial?" asked Luna.

"That means you write a polynomial as a product of like this

$$p(x) = a_0 + a_1\, x + a_2\, x^2 \ldots = c\,(x - \lambda_0)\,(x - \lambda_1)\,(x - \lambda_2)\ldots \quad , \tag{5.2}$$

where c is an overall factor and the λ_i are the roots to the polynomial. That's where $p(x) = 0$."

Luna gave Twilight a look that seemed like a combination of disgust and pity. "You really do not know this material at all, do you?" she said.

"But I thought—"

"I strongly doubt you were, Twilight Sparkle! For one thing, is the polynomial defined over the real numbers or the complex numbers, or something else entirely?"

"Well I didn't—"

"And is the polynomial finite or infinite?"

"I didn't realize—"

Theorem 5.1

"Indeed. The *correct* answer is to say that a *finite* polynomial can be factored this way *if* it is over an algebraically closed field:

$$c\left(\left(x-\lambda_0\right)\left(x-\lambda_1\right)\ldots\right)=0 \qquad , \tag{5.3}$$

Twilight frowned in frustration. She could never win with Luna. “What, pray tell, is an algebraically closed field?”

“An algebraically closed field means that every finite polynomial in the field has a root. Not all polynomials in every field can be factored. For example, just think of the polynomial

$$1+x^2=0 \qquad .\text{”} \tag{5.4}$$

“Its solutions are i and $-i$,” said Twilight.

Luna shook her head. “No no no. Do not skip ahead! We have not said yet what field this polynomial is defined over.”

“Well what is it then?” said Twilight impatiently.

Theorem 5.2

“If this polynomial were defined over the complex numbers, then yes, its solutions would be i and $-i$. If it were defined over the real numbers, it would have no solutions. The *fundamental theorem of algebra* says that the complex numbers are, in fact, an algebraically closed field. We *may* get to a proof of it eventually, but for now, you should just trust me on it.”

Twilight nodded.

“The first thing to prove about factoring polynomials is to relate a root to a factor. Suppose a polynomial $p(x)$ has a linear factor $x-\lambda$, or, in other words,

$$p(x)=(x-\lambda)\,p'(x) \qquad . \tag{5.5}$$

Clearly λ is a root of $p(x)$.”

Lemma 5.3

Luna continued, “But what about the converse? Suppose λ is a root of $p(x)$, or, in other words and by definition, $p(\lambda)=0$. Can a linear polynomial always be factored out? To prove this, we need a tedious little algebraic lemma.” Luna wrote

$$x^n-\lambda^n=(x-\lambda)\left(x^{n-1}\lambda^0+x^{n-2}\lambda^1+\ldots+x^1\lambda^{n-2}+x^0\lambda^{n-1}\right) \tag{5.6}$$

"To prove this, write it as a summand and then expand like so:"

$$\begin{aligned} x^n - \lambda^n &= (x-\lambda)\left(x^{n-1}\,\lambda^0 + x^{n-2}\,\lambda^1 + \ldots + x^1\,\lambda^{n-2} + x^0\,\lambda^{n-1}\right) \\ &= (x-\lambda) \sum_{0\le i<n} x^i\,\lambda^{n-i-1} \\ &= \sum_{0\le i<n} \left(x^{i+1}\,\lambda^{n-i-1} - x^i\,\lambda^{n-i}\right) \end{aligned} \tag{5.7}$$

The next part is some annoying summand algebra." Luna wrote

$$\begin{aligned} x^n - \lambda^n &= (x-\lambda)\left(x^{n-1}\,\lambda^0 + x^{n-2}\,\lambda^1 + \ldots + x^1\,\lambda^{n-2} + x^0\,\lambda^{n-1}\right) \\ &= (x-\lambda) \sum_{0\le i<n} x^i\,\lambda^{n-i-1} \\ &= \sum_{0\le i<n} \left(x^{i+1}\,\lambda^{n-i-1} - x^i\,\lambda^{n-i}\right) \end{aligned} \tag{5.8}$$

"To do the next step, notice that the second part of every term in this summand is the opposite of the first part of the following term. This means that the entire summand cancels out except for the first part of the first term and the second part of the last term. We end up with

$$= x^n - \lambda^n + \sum_{1\le i<n} \left(x^{i+1}\,\lambda^{n-i-1} + x^{i+y}\,\lambda^{n-i-1}\right) = x^n - \lambda^n \qquad ." \tag{5.9}$$

Twilight nodded, trying to follow along.

Proposition 5.4

"Now that the lemma is done, we can get on to the main result. Let it be given that λ is a root of $p(x)$, which means that $p(\lambda) = 0$. Then we can write

$$p(x) = p(x) - p(\lambda) = a_1\,(x-\lambda) + a_2\left(x^2 - \lambda^2\right) + a_3\left(x^3 - \lambda^3\right) + \ldots \tag{5.10}$$

Lemma 5.3 says that a linear polynomial can be factored out of each of these terms. We can write expression 5.5 as

$$\begin{aligned} p(x) &= a_1\,(x-\lambda) + a_2\,(x-\lambda)\,p_1(x) + a_3\,(x-\lambda)\,p_2(x) + \ldots \\ &= (x-\lambda)\,p'(x) \end{aligned} \tag{5.11}$$

In other words, if a polynomial has a root, than a linear polynomial can be factored out of it. The details of the polynomial $p'(x)$ do not matter, other than to note that its degree must be less than that of $p(x)$. The reason this is important is that proposition 5.4 can be applied recursively, and we can know that the resulting polynomial is simpler with each application.

Proposition 5.5

If the polynomial is finite, and if the polynomial is defined over an algebraically closed field, we can continue to factor out linear polynomials until only a linear polynomial is left. The linear polynomial that is left at the end might have an overall scalar factor that can be factored out as well. Hence

$$c\left(\left(x-\lambda_0\right)\left(x-\lambda_1\right)\ldots\right)=0 \qquad .\text{”} \tag{5.12}$$

“Just what I wrote earlier.” said Twilight with some irritation.

“This is true for every algebraically–closed field. For fields that are not algebraically closed, there is no general rule about how polynomials factor. For example, the rational numbers are not an algebraically closed field, and it is easy to construct polynomials in it that cannot be factored at all. For example, $1+2\,x+x^3=0$ is a rational polynomial but it has no roots among the rationals.

Theorem 5.6

However, there *is* a rule for factoring polynomials over the real numbers. The polynomial will factor into something like this:

$$\begin{gathered} c\left(\left(x-I\,\lambda_0\right)\left(x-I\,\lambda_1\right)\ldots\left(x^2-x\left(\mu_0+\mu_0^*\right)-\mu_0\,\mu_0^*\right)\ldots\right) \\ \left(x^2-x\left(\mu_1+\mu_1^*\right)-\mu_1\,\mu_1^*\right)\ldots\right)=0 \qquad . \end{gathered} \tag{5.13}$$

In this expression, there are two kinds of roots. Real roots λ_i and complex pairs of roots μ_i and μ_i^*.

Proposition 5.7

If a complex number is a root of a real polynomial, so is its complex conjugate. Suppose a polynomial has real coefficients and it has a root λ. Then

$$0=\left(a_0+a_1\,\lambda+a_2\,\lambda^2\ldots\right)^*=a_0+a_1\,\lambda^*+a_2\,\lambda^{*2}\ldots \tag{5.14}$$

which proves that λ^* is a root as well. If λ is a real number than this tells us nothing, but if it is complex it gives us a totally different complex root!

“And that actually holds for infinite polynomials too,” said Twilight.

“Why yes, I suppose it does! Since the fundamental theorem of algebra proves to us that we can factor polynomials into linear polynomial factors, we can write

$$p(x) = a_0 + a_1\, x + a_2\, x^2 \ldots = \\ (x - \lambda)\,(x - \lambda^*)\, p_{n-2}(x) = \left(x^2 - (\lambda + \lambda^*)\, x + \lambda\, \lambda^*\right) p_{n-2}(x) \tag{5.15}$$

But the factor $x^2 - (\lambda + \lambda^*)\, x + \lambda\, \lambda^*$ is actually a quadratic real polynomial.

Proposition 5.8

Now we can finally get expression 5.13. First, factor out all the real roots. Then factor out the pairs of complex roots as quadratic polynomials. Then factor out any remaining overall scalar factor and you're done!"

"...And why do I need to know any of this again?"

"You will understand very soon, Twilight Sparkle. Possibly tomorrow. But one thing I can tell you now is that we will find a correspondence between invertible linear operators and polynomials. We will be able to use this to show that every rotation operator should, in some sense, factor like a real polynomial. This will complete the proof that

"Alright Princess. Very well then."

"Now, what about factoring infinite polynomials? I have already pointed out that the proofs I gave do not work, but do you know of any infinite polynomials which would provide a counterexample?"

"No."

"I can give you a good one. Consider the infinite polynomial

$$f = 1 + x + x^2 + x^3 + x^4 \ldots \tag{5.16}$$

Does this polynomial have any roots?"

"I don't know, Princess."

"No, it does not. You can see this by observing that

$$f = 1 + x + x^2 \ldots = 1 + x\left(1 + x + x^2 \ldots\right) = 1 + x\, f \tag{5.17}$$

"Solve for f and you get $f = \frac{1}{1-x}$. Of course that only works when the value of f is finite. That is, when $x < 1$. Otherwise, f is infinite because every term in the series is larger than the previous."

"Of course," said Twilight, barely following.

"But you can see that the expression $\frac{1}{1-x}$ is never zero for $x < 1$. Thus, f is either infinite or it is a finite expression that is never zero. That means no roots."

"Alright then," said Twilight to sum up. "A polynomial factors differently depending on whether it is finite or infinite and whether it is real or complex."

"Quite so," said Luna. "Well that's the most important part of today's lesson. I do not think we have enough time to start something new, so perhaps we should conclude with something fun!"

"Perhaps instead I could leave early? I mean there's really a lot of work I should be..." Twilight paused when she saw Luna's crestfallen expression. "I mean... I suppose I have time for a bit of fun..."

"Oh excellent. We don't often have time to do any *pure* math around here."

"Isn't that the *only* thing we've done so far?"

"I have already told you that everything we have ever done will be critical to your understanding of physics. But now I think we will do just a bit of pure mathematics. I'm sure you know the quadratic formula of course, but do you know the formulas to solve cubic and quartic polynomials? I will show you the derivations."

"Um, yes that sounds like *great* fun, Princess."

"I'm glad you think so! Let's start with a linear polynomial. How would you solve something like this?"

$$a + b\,x = 0 \qquad (5.18)$$

"Um, that one's kind of obvious, Princess. I think I could do that one when I was a filly. It's just

Theorem 5.9

$$x = -\frac{a}{b} \qquad .” \tag{5.19}$$

“Yes well… we just included that for completeness. Now on to the quadratic polynomial.”

“Also one I already know,” said Twilight.

$$a + b\,x + c\,x^2 = 0 \tag{5.20}$$

“Let me show you a trick to simplify first. We shall normalize the polynomial by dividing by c. This removes the factor on x^2. Then the polynomial becomes

$$a' + b'\,x + x^2 = 0 \tag{5.21}$$

where $a' = \frac{a}{c}$ and $b' = \frac{b}{c}$. We will need that trick in the rest of the derivations, so get used to it!”

“...Alright.” said Twilight. “Then I just complete the square

$$a' + b'\,x + x^2 = a' - \frac{b'^2}{4} + \left(\frac{b'}{2} + x\right)^2 = 0 \qquad , \tag{5.22}$$

Theorem 5.10

and then I can then solve for x again.

$$x = \pm\sqrt{a' - \frac{b'^2}{4}} - \frac{b'}{2} \tag{5.23}$$

There are two solutions because the square root could be positive or negative.”

“Quite right, but I must object to your use of the ± symbol. This symbol really on makes sense when you are working with real numbers. A positive real number has two square roots, one positive and one negative, and a negative real number has two square roots, one positive imaginary and one negative imaginary. However, for complex numbers generally, these two roots get mixed up and there isn’t one that you can say is objectively the positive or negative one. It makes more sense to write something like this:”

$$x = \overbrace{\sqrt{a' - \frac{b'^2}{4}}}^{\text{both}} - \frac{b'}{2} \tag{5.24}$$

"What do you mean that the roots get mixed up?"

"Hmm. Imagine for a moment," said Luna, "a complex number $p \equiv (\cos(\theta) + i\sin(\theta))\,|\,p\,|$, where $|\,p\,|$ is a real number. The parameter θ rotates p around a circle centered at zero. What do you think happens to the square roots of p as it rotates around?"

"I don't know. Maybe there is a way to find an expression for the square root of p?"

"Here's a hint. Use the half-angle formulas!"

"Oh, I think I see. I can just make the replacements $\cos(\theta) \to \cos\left(\frac{\theta}{2}\right)^2 - \sin\left(\frac{\theta}{2}\right)^2$ and $\sin(\theta) \to 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$ and I get a nice perfect square." Twilight wrote

$$\begin{aligned} p &= (\cos(\theta) + i\sin(\theta))\,\Big|\,p\,\Big| \\ &= \left(\cos\left(\frac{\theta}{2}\right)^2 + 2\,i\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)^2\right)\Big|\,p\,\Big| \\ &= \left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right)^2\Big|\,p\,\Big| \quad . \end{aligned} \tag{5.25}$$

"So," Luna interrupted, "the square roots of p are

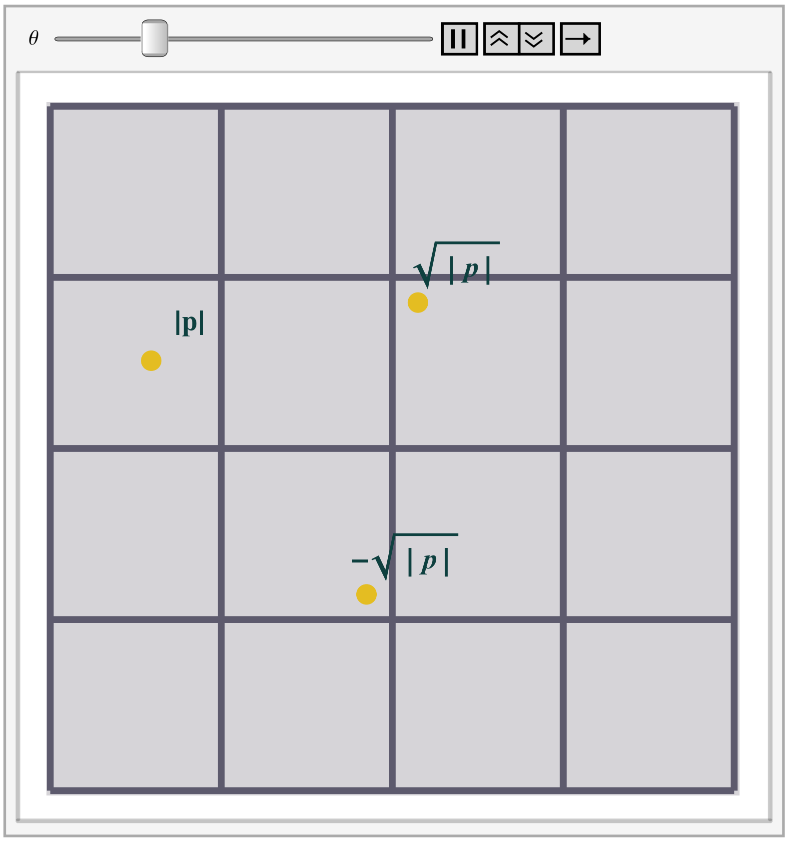
$$\sqrt{p} = \left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right)\sqrt{|\,p\,|} \quad , \tag{5.26}$$

where $\sqrt{|\,p\,|}$ is the positive square root of $|\,p\,|$. Since $|\,p\,|$ is real, there is, objectively, one positive root so there isn't a problem in this case. We can get both roots by observing that since θ and $\theta + 2\pi$ are equivalent, you could interpret $\frac{\theta}{2}$ as either itself or as $\frac{\theta}{2} + \pi$."

"Or just by writing ±," grumbled Twilight under her breath.

Twilight continued. “So this expression says that as you rotate p around a circle, the square roots of p rotate at half the rate.”

“Indeed. Good observation, Twilight Sparkle! This means that if p goes all the way around the circle, the square roots of p have only rotated half way around. Thus, what was originally the positive square root of p has become the negative one and vice versa.”



“I never noticed that before, Princess!”

“This is why there is no objectively positive or negative square root of a complex number. It is arbitrary to specify one or the other. There is a similar relation for higher roots. For example, there are three cube roots for a complex number, arranged in an equilateral triangle. If you rotate a complex number around the origin, its cube roots rotate a third of the way around. Its four tesseract roots rotate one fourth of the way around and so on. ...Oh dear.”

“What is it, Princess?”

“I just realized that you probably will want to know about these properties of complex numbers for doing physics! That means this isn't a *total* digression."

Twilight rolled her eyes a little. "Don't worry Princess. I think I can handle it."

"Alright, but some day we must do some really, really *pure* mathematics. But now I want to show you another trick. Going back to the polynomial

$$a' + b'\,x + x^2 = 0 \qquad , \tag{5.27}$$

try making the substitution

$$x \to x' - \frac{b'}{2} \qquad . \tag{5.28}$$

This gives

$$\begin{aligned} a' + b'\,x + x^2 &= \\ a' + b'\left(x' - \frac{b'}{2}\right) + \left(x' - \frac{b'}{2}\right)^2 &= \\ a' + \left(b'\,x' - \frac{b'^2}{2}\right) + \left(x'^2 + \frac{b'^2}{4} - b'\,x'\right) &= \\ a' - \frac{b'^2}{4} + x'^2 &= 0 \qquad . \end{aligned} \tag{5.29}$$

Thus, we have eliminated the linear term from the polynomial and are left with something that can easily be solved. You see?

$$x' = \sqrt{a' - \frac{b'^2}{4}} \tag{5.30}$$

To get the more general solution, simply replace back in the definitions of x', a' and b'. There is a similar trick for all higher-order polynomials."

Twilight nodded.

"Next, finally we shall do something new. The cubic equation! I write it already normalized."

$$a + b\,x + c\,x^2 + x^3 = 0 \tag{5.31}$$

Now, with the replacement

$$x \to x' - \frac{c}{3} \quad , \tag{5.32}$$

we get a simpler cubic with the quadratic term eliminated.

$$a'' + b''\,x' + x'^3 = 0 \tag{5.33}$$

What does this do for us? Look at the condition that is imposed on the solution of this equation by removing the quadratic term. Let us write a cubic polynomial that has been factored into its three roots λ_0, λ_1, and λ_2. Then if we expand that out like this

$$\begin{gathered}(\lambda_0 - x)\,(\lambda_1 - x)\,(\lambda_2 - x) = \\ \lambda_0\,\lambda_1\,\lambda_2 - (\lambda_0\,\lambda_1 + \lambda_0\,\lambda_2 + \lambda_1\,\lambda_2)\,x + (\lambda_0 + \lambda_1 + \lambda_2)\,x^2 - x^3 \quad ,\end{gathered} \tag{5.34}$$

you can see that if the quadratic term (which I've highlighted in pink) is to be zero, the solutions to the polynomial must all sum to zero. What can we do with this knowledge?"

"I don't know. Effectively it means there are only two solutions to find because the third is determined by the others."

Definition 5.1 : *Ansatz*

"Right, and there is a clever trick to take advantage of that. Now if you wanted *two* independent numbers whose sum was p, you could introduce the a number q and just arbitrarily say that the numbers you wanted are $p + q$ and $p - q$. You could write every possible pair of numbers this way, so you can always declare that they will come in that form. When you declare that you will write everything in a particular form without loss of generality, this is called an *ansatz*. The next step in the derivation of the cubic is to come up with the right ansatz.

Now, take the expression $\sqrt[3]{p}$, where p is a complex number. The cube root of p actually stands for three complex numbers which form an equilateral triangle centered at the origin."

"So they will sum to zero."

“Indeed. You are thinking along the right lines. The trick is to write x as $\sqrt[3]{p} + \sqrt[3]{q}$, the sum of two cube roots, and to show that this is an ansatz for three complex numbers whose sum if zero.”

Newly sensitized to the subtleties of complex roots, Twilight asked, “what does that expression mean exactly? Is each cube root added to each of the others?”

Luna leaned very close to Twilight’s face. “Good question. You are really starting to understand what you are dealing with. But if you added each root of p to each of q, you would have nine numbers, and we only need three. Instead, you only add one cube root of p to a cube root of q and then a different cube root of p to a different cube root of q until you have run out.”

“So we can definitely say that the sum of all the numbers represented by $\sqrt[3]{p} + \sqrt[3]{q}$ is zero, since each set of cube roots sums to zero,” said Twilight. “But how do we choose which ones to add?”

“Don’t worry about that question for now. I will first prove that the ansatz works. Let us say that we want to make the complex numbers u, v, and $-u-v$ out of $\sqrt[3]{p} + \sqrt[3]{q}$, and let us write the cube roots of p and q as p_i and q_i, where $0 \le i \le 2$. We want to be able to say that

$$\begin{aligned} u &= p_0 + q_0 \\ v &= p_1 + q_1 \\ -u - v &= p_2 + q_2 \end{aligned} \quad . \tag{5.35}$$

We know that the cube roots of p and q form equilateral triangles centered at the origin, so we can say that they are numbered either clockwise or counterclockwise. If the labels of both sets of roots go in the same direction, then v is constrained by the value of u. It is just a $120\,°$ rotation of u, either clockwise or counterclockwise. If that is the case, then $\sqrt[3]{p} + \sqrt[3]{q}$ just represents another equilateral triangle centered about the origin, which does not help us one bit. Therefore, we must say that the roots of p and q are labeled in opposite directions. Let us just arbitrarily say that q goes counterclockwise and p goes clockwise. Then we can write

$$\begin{aligned} u &= p_0 + q_0 \\ v &= \frac{1}{2}\left(1 + i\sqrt{3}\right)p_0 + \frac{1}{2}\left(-1 + i\sqrt{3}\right)q_0 \end{aligned} \qquad . \tag{5.36}$$

You can think of (p_0, q_0) and (u, v) each as spanning sets of the complex plane viewed as a two–dimensional vector space. The transformation between them is the matrix

$$\begin{pmatrix} 1 & 1 \\ \frac{1}{2}\left(1 + i\sqrt{3}\right) & \frac{1}{2}\left(-1 + i\sqrt{3}\right) \end{pmatrix} \qquad . \tag{5.37}$$

This is an invertible matrix, which means that for any p_0 and q_0 you can always find a u and v and vice versa."

"Thus proving the ansatz!" concluded Twilight, "But which two roots do we add?"

"Actually, it does not matter. Since the cube roots are arbitrary, it does not matter which root of p and q you begin with. You must ultimately get the same answer either way."

"What is the proof of that?"

"It already *is* proven *because* the roots are arbitrary. Necessarily it cannot matter. If you want to convince yourself of that, you can try adding cube roots together on your own time."

"Very well, Princess."

"We are almost done with the cubic now," continued Luna. "We substitute the new expression in for x and get

$$\begin{gathered} a'' + b''\,x' + x'^3 = \\ a'' + b''\left(\sqrt[3]{p} + \sqrt[3]{q}\right) + \left(\sqrt[3]{p} + \sqrt[3]{q}\right)^3 = \\ (a'' + p + q) + \left(b'' + 3\sqrt[3]{p}\,\sqrt[3]{q}\right)\left(\sqrt[3]{p} + \sqrt[3]{q}\right) = 0 \end{gathered} \qquad . \tag{5.38}$$

In order for this to be true, it must be that both terms on the left are zero. Do you see why?

"Because a'' and b'' are arbitrary. If the two terms had to sum to zero without both being zero individually, this would put a condition on a'' and b''. We need to be able to write a'' and b'' as separate expressions of p and q."

"That is correct. So that gives the two equations

$$\begin{aligned} a'' &= -p-q \\ b'' &= -3\sqrt[3]{p}\,\sqrt[3]{q} \end{aligned} \tag{5.39}$$

which simplify to

$$-\frac{b''^3}{27} - a''\,p + p^2 = 0 \qquad . \tag{5.40}$$

or

$$-\frac{b''^3}{27} - a''\,q + q^2 = 0 \qquad . \tag{5.41}$$

These are quadratic equations. In fact, they are both the *same* quadratic equation, but with a different variable. However, if you let p be equal to one root, you find that q must be equal to the other root. So we can then write

$$\begin{aligned} p &= \frac{1}{2}\left(a'' + \overset{\text{any}}{\sqrt{a''^2 + \left(\frac{2}{3}\right)^2 b''^3}} \right) \\ q &= \frac{1}{2}\left(a'' - \overset{\text{any}}{\sqrt{a''^2 + \left(\frac{2}{3}\right)^2 b''^3}} \right) \qquad , \end{aligned} \tag{5.42}$$

as long as you pick the same root in both expressions. Then the solution to the cubic is

$$\overset{\text{clockwise}}{\sqrt[3]{\frac{1}{2}(a''+d)}} + \overset{\text{counterclockwise}}{\sqrt[3]{\frac{1}{2}(a''-d)}} \qquad , \tag{5.43}$$

where

$$d = \sqrt[\text{any}]{a^2 + \left(\frac{2}{3}\right)^2 b''^3} \qquad . \tag{5.44}$$

“That was lovely, Princess.”

“Thank you, Twilight Sparkle,” Luna actually smiled at that. “Now on to the quartic equation. The quartic can be simplified by some similar steps to the ones we used with the cubic, so we shall start out with

$$a + b\,x + c\,x^2 + x^4 = 0 \qquad . \tag{5.45}$$

Next, add $c\,x^2 + c^2$ to both sides of the equation to complete the square on one side.

$$\begin{aligned} c\,x^2 + x^4 &= -a - b\,x \\ c^2 + 2\,c\,x^2 + x^4 &= c^2 - a - b\,x + c\,x^2 \\ \left(c + x^2\right)^2 &= c^2 - a - b\,x + c\,x^2 \end{aligned} \tag{5.46}$$

That was a good trick, but here is where the real trick comes in. We want to be able to make *both* sides of the equation into perfect squares. To do this, change the left side of the equation to $\left(c + x^2 + \Xi\right)^2$. If you expand that out, you can see that this is equivalent to adding $2\,c\,\Xi + 2\,x^2\,\Xi + \Xi^2$. So,

$$\begin{aligned} \left(c + x^2 + \Xi^2\right)^2 &= c^2 + 2\,c\,\Xi + \Xi^2 - a - b\,x + (c + 2\,\Xi)\,x^2 \\ &= (c + \Xi)^2 - a - b\,x + (c + 2\,\Xi)\,x^2 \qquad . \end{aligned} \tag{5.47}$$

If the right side is a perfect square in x, it must be that

$$-b = 2\sqrt{(c + \Xi)^2 - a}\;\sqrt{c + 2\,\Xi} \qquad . \tag{5.48}$$

or

$$\begin{aligned} \frac{b^2}{2} &= \left((c + \Xi)^2 - a\right)(c + 2\,\Xi) = \\ &\left(-a\,c + c^3\right) - 2\,a\,\Xi + 4\,c^2\,\Xi + 5\,c\,\Xi^2 + 2\,\Xi^3 \qquad . \end{aligned} \tag{5.49}$$

This is a cubic equation in Ξ! Something you just learned to do. It does not matter which solution you choose because each is able to form a perfect square out of expression 5.47 and each gives the same solutions. The expression then becomes

$$\left(c + x^2 + \Xi^2\right)^2 = (c + \Xi)^2 - a - b\,x + (c + 2\,\Xi)\,x^2$$
$$\left(c + x^2 + \Xi^2\right)^2 = \left(\overset{\text{any}}{\sqrt{(c + \Xi)^2 - a}} + \overset{\text{any}}{\sqrt{c + 2\,\Xi}}\;x\right)^2 \qquad . \tag{5.50}$$

Here you have to choose which square roots you want to use. You may choose whichever you like and the result will still work, but notice that in expression 5.48 you already made a choice about the products of the roots. So make sure your choice is consistent with that!

Theorem 5.11

Next, take the square root of this expression and get a quadratic in x:

$$s_0\left(c + x^2 + \Xi^2\right) = \overset{\text{any}}{\sqrt{\left(c + \Xi^2\right)^2 - a}} + \overset{\text{any}}{\sqrt{c + 2\,\Xi}}\;x \qquad . \tag{5.51}$$

Here we make a choice that is not arbitrary. You could have chosen either square root and each will give different answers. I have added the factor s_0 to represent this choice. Any solution you can find when s_0 is 1 must also give solutions when is factor that represents the choice you made when you took the square root. The solutions whose solutions are

$$x = \frac{1}{2\,s_0}\left(A + s_1\,\sqrt{A^2 + 4\left(s_0\,B - 4\,c - 4\,\Xi^2\right)}\right) \qquad , \tag{5.52}$$

where s_1 is the choice of solution

$$A = \overset{\text{any}}{\sqrt{c + 2\,\Xi}}$$
$$B = \overset{\text{any}}{\sqrt{-a + \left(c + \Xi^2\right)^2}} \qquad . \tag{5.53}$$

and Ξ is a solution to the cubic equation given before.”

Twilight nodded and wondered how long this progression would continue before she would be set free.

“Now as to the quintic equation—

“Um, Princess. Don’t you think we’re a little overtime already?”

“Yes we are. Too bad, because the quintic equation is where it *really* starts to get interesting.”

“It does?”

“It turns out that there is *no* general solution to quintic polynomials that can be written using only sums, products, and roots. To prove that requires going into some very interesting and advanced mathematics.”

“Really?”

“Unfortunately, to understand this requires *much* more than what we can go into tonight. Or ever!“

“Oh…” said Twilight, half disappointed and half relieved.

“Now off with you, disciple!”