Theoretical Physics is Magic

Smeapancol

Unit 1: Space

Lesson 1- Algebra

Day 4- Rotations

The next day Twilight found Luna sitting quietly just inside the gate to the Canterlot gardens.

“Good morning, Princess.”

“Good morning, Twilight Sparkle. Today we shall talk more about transformations.”

Twilight grinned. “You mean like how you transformed into Nightmare Moon?”

Luna pouted slightly. “No, foolish Twilight Sparkle! I mean something much more interesting: *coordinate* transformations!”

Twilight could see that she had said the wrong thing. “I'm sorry, Princess! Just a joke, haha!”

“Hmf! Very well.” Luna began to lead her to the Canterlot wall, where a green pegasus colt with a sponge was cleaning their work from yesterday.

Luna waited for him to clear off enough room on the wall for them to begin work. Then her horn glowed a moment and the pegassus disappeared with a zap.

“Um.” said Twilight. “Where did he go?”

“Oh, I just sent him to somewhere else where there is a mess.”

“I don’t think you should just teleport people away without warning.”

“Why not? He’s my employee, so I can do whatever I want with him! Plus, it’s more efficient that way.”

“But that might terrify him! Don’t you think that could be disconcerting.”

Luna grumbled. “Well maybe. But I didn’t know what to say to him!”

Twilight sighed. “Well maybe you could say, ‘Thank you, and if you wouldn’t mind just find something else to do until we’re done here.’”

Luna shook her haid. “No no no! That sounds simply awful!”

“But—”

Definition 4.1 : *Coordinate Transform*

“No no no, I said. Anyway, to transform the coordinates is to define a new set of coordinates and transform everything so that they are described in the new set of coordinates instead of the old ones. For a vector space, that just means a change of basis. We know that there is one linear map for every basis map and that a map from one basis to another is invertible. So therefore we define a coordinate transform is an invertible operator.”

“Now look, you did it again!” said Twilight now a little irritated. “You took something that is *meaningful* in our intuition, in this case a coordinate transform, mentioned some properties that it has, and then redefined it in terms of those properties, thus turning it into something *meaningless*.”

Luna considered that for a moment. “Philosophically, we start by using our intuition. Intuition has rules that make it work, but they are not conscious. So we have to search for rules that predict our intuition. Once we have those rules, then we rely on the rules, not our intuition.”

“To change coordinates by a transform T, you multiply a vector by the operator, yes?”

“Yes.”

"But what do you do to change the coordinates of a linear map?"

"I'm not sure."

"Well let us think about it. If $\vec{w} = B\,\vec{v}$, then a change of coordinates requires that

$$\begin{aligned} T\,\vec{w} &= B'\,T\,\vec{v} \\ \vec{w} &= T^{-1}\,B'\,T\,\vec{v} \end{aligned} \tag{4.1}$$

where B' is the coordinate transformed version of B. To make this work, we need that $B = T^{-1}\,B'\,T$, or in other words that $B' = T\,B\,T^{-1}$. This is how you do a coordinate transform on a linear map. Understand?"

"I understand, Princess!" Twilight said.

"This is another thing I want you to be able to think about in two different ways. You may have already realized this, but a linear map *is* a vector. It can be added to other maps and multiplied by scalars, so necessarily so. An operator B in the vector space V is a vector in the space $V \times V$."

"What does the cross mean, Princess?"

"That is the Cartesian product. The Cartesian product operates on sets which have similar operations defined on them and it allows those operations to distribute over the product."

"Ummmm… what?"

"I mean, suppose A and B are both sets with addition defined on them, and suppose A has members a_i and B has members b_i. Then $\{a_i, b_i\} \in A \times B$ for all i. You see? Each element of $A \times B$ is just takes one element from A and one element from B. Addition is then defined by $\{a_1, b_1\} + \{a_2, b_2\} = \{a_1 + a_2, b_1 + b_2\}$."

"So $V \times V$ is another vector space?"

"Correct. Although it's not always the case that the Cartesian product of two things is always the same kind of thing. For example, the Cartesian product of two fields is a ring."

"Ok!"

"Now look again at the expression $T\,B\,T^{-1}$. It *looks* like a linear map multiplied on the right and left by other linear maps so as to apply a coordinate transform, yes?"

Twilight nodded.

"But look at the following equation. What does it suggest to you?"

$$T\,(b\,B + c\,C)\,T^{-1} = b\,T\,B\,T^{-1} + c\,T\,C\,T^{-1} \tag{4.2}$$

"It looks a lot like the rule defining a linear map."

"Actually, it looks *exactly* like the rule defining a linear map! This proves that T can be considered a linear operator on $V \times V$, where multiplication by a vector B is defined as a coordinate transform on the operator B. So when you see $T\,B\,T^{-1}$, I want you to think, this is both a coordinate transform, but it is also $T\,\vec{B}$."

"Got it, Princess!"

"Now I will show that if V is an inner product space, then $V \times V$ is also an inner product space. It's really pretty clear now we want to be able to write something that is equivalent to $\vec{B}\,g\,\vec{C}$. You know how to apply g to C by performing a coordinate change to get $g\,C\,g^{-1}$. Next we would have to do something like a dot product. We would want simply to sum over all the corresponding components of the matrices for B and C. It turns out you can do that this way:

$$\vec{B} \cdot \vec{C} = \mathrm{Tr}\left(B^T\,g\,C\,g^{-1}\right) \tag{4.3}$$

Definition 4.2 : *Trace*

where the Tr stands for the *trace*. That means sum along the diagonal of a matrix, so

$$\mathrm{Tr}(B) = B_{aa} \tag{4.4}$$

I suppose we shall prove that expression 4.3 is an inner product. How would you do that?

Twilight replied, "The expression is linear, so that is one property. The other property is symmetry. To make it easier, I can write expression 4.3 without the gs and just say that we're in an orthonormal basis."

Luna nodded. "Good start. Next try abstract index notation."

Twilight didn't think she needed a hint but she didn't say anything.

$$\mathrm{Tr}\left(B^T\, C\right) = B_{ab}\, C_{ab} = \mathrm{Tr}\left(C^T\, B\right) = \mathrm{Tr}\left(C\, B^T\right) \tag{4.5}$$

"Well that's clearly symmetric." she said.

"Yes, and what is the lesson? There are many more vectors sitting around than just the spatial vectors and you should never listen to any ignoramus who says a vector is something with a magnitude and direction!"

Twilight could not help but roll her eyes.

"Now," Luna continued, "we shall think about a coordinate transform O that preserves the inner product. Since the inner product defines the concept of an angle between two vectors, this is the same as saying that O applied to any two vectors must preserve the angle between any two vectors it is applied to, right?"

"Yes."

Definition 4.3 : *Orthogonal Transformation*

"The O stands for *orthogonal transformation*, which is what this is. That means that $O\, g\, O^{-1} = g$, or that O and g commute with one another. These transforms are very special! Orthogonal transformations (and their generalizations) form the basis for all physics! So pay attention!" Luna stomped loudly.

Twilight started at the sharp sound of Luna's palladium shoes against the stone. "I'm listening!"

"Start with a general inner product expression, and then perform a coordinate transform.

$$\vec{y} \cdot \vec{x} = \vec{x}\, g\, \vec{y} = \vec{x}\, O^T\, O\, g\, O^{-1}\, O\, \vec{y} = \vec{x}\, O^T\, g\, O\, \vec{y} \tag{4.6}$$

Since g commutes with O, it follows that

$$\begin{aligned} O^T\, g\, O &= g \\ O^T\, O\, g &= g \\ O^T\, O &= 1 \end{aligned} \tag{4.7}$$

This uses the fact that g is invertible."

Theorem 4.1

"So," said Twilight, "a transformation that preserves the cross product is the inverse of its transpose. Or, in other words, to invert such an operator, merely take the transpose."

"Exactly! It is also the case that the product of orthogonal transformations is an orthogonal transformation.

Proposition 4.2

$$O_1\,O_2\,(O_1\,O_2)^T = O_1\,O_2\,O_2^T\,O_1^T = 1 \tag{4.8}$$

Since 1 is an orthogonal transformation, this makes orthogonal transformations a group, something you may recall we briefly mentioned on day 1. We shall have more to say about that later!"

Luna sniffed a little. "Do you remember yesterday when we came up with a formula for a projection operator perpendicular to a unit vector? it was

$$P_\perp(\vec{p}) = 1 - \vec{p} \otimes \vec{p}$$."

Twilight nodded.

"You must imagine a similar operator which performs a flip, which we will write as $F_{\vec{p}}$. It should invert vectors parallel to $\vec{p}$ and leave other vectors the same."

"Yes, Princess! I can do that! See, here it is already."

$$F_{\vec{p}} = 1 - 2\,\vec{p} \otimes \vec{p}$$

Definition 4.4 : *Vector Flip*

"Yes! That's a kind of flip. You just go twice as far as the projection and you end up with an inversion. This kind of flip I'll call a *vector flip* because it's defined as a flip over a single vector.

Definition 4.5 : *Flip*

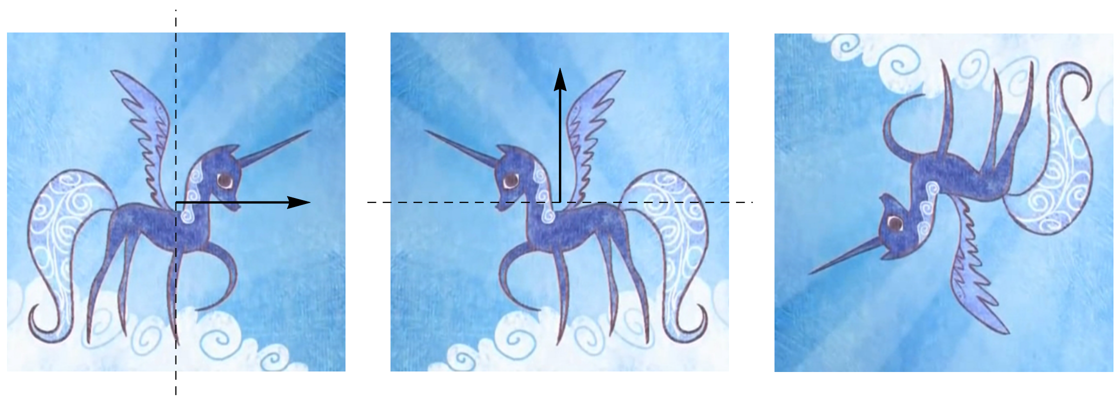
A more general idea of a *flip* is an orthogonal transformation that is its own inverse. It is easy to see that $F_{\vec{p}}$ has this property. The flip inverts one component of the vector, so the same flip again will revert it to how it was. It also has a symmetric matrix, so it is an orthogonal transformation. Can you think of any other operations with this property?"

"A 180 degree rotation?"

"That is true! Of course, a 180 degree rotation is the product of two perpendicular flips, correct?"

"Is it Princess?"

"Yes! Give it a try with this picture. First flip across $(1, 0)$, and then flip across $(0, 1)$. See?"



Twilight played with the picture. "I understand the example, Princess, but is this true in general?"

"What do you think, Twilight Sparkle?"

"Come to think of it, that reminds me of a question I had yesterday. Each of the two flips picks out a particular vector, but the final result, the 180° rotation, does not. The only thing that is specially picked out is the plane of rotation. So as long as both flips were in the same plane and were perpendicular to one another, the final result would have to be the same 180° rotation. We can have the flips aligned to the axes without limiting our general knowledge of the resulting rotation."

"Yes! Exactly that! You are really learning now, Twilight Sparkle. That was a very good symmetry argument."

Twilight blushed a little at the praise.

Luna continued. "I will call operators like $F_{\vec{p}}$ a one-dimensional flip because it only flips one vector. A 180° rotation is a two-dimensional flip. You could have flips for any number of dimensions. Are there any other possibilities for one–dimensional flips other than $F_{\vec{p}}$?"

"The identity operator could be considered a one–dimensional flip."

"Right. There are no other possibilities for a one-dimensional flip," Luna continued. "This can be seen by thinking of flips as operators acting on a given one–dimensional subspace. A one–dimensional real vector space is just the real numbers, and you know that -1 and 1 are the only numbers which are their own inverses.

Thinking about flips more generally, suppose that there were a general flip f, and let's say that $f\,\vec{v} = \vec{w}$. Then since by definition $f^2\,\vec{v} = \vec{v}$, it must be that $f\,\vec{w} = \vec{v}$. Now therefore

$$\begin{aligned} f\,(\vec{v} + \vec{w}) &= \vec{v} + \vec{w} \\ f\,(\vec{v} - \vec{w}) &= -(\vec{v} - \vec{w}) \end{aligned} \tag{4.9}$$

There are two possible ways to interpret this. If $\vec{v}$ and $\vec{w}$ are linearly independent, then f acts on the space they span as $F_{\vec{v}-\vec{w}}$. On the other hand, if $\vec{v}$ and $\vec{w}$ are not linearly independent, then $\vec{w} = \pm\vec{v}$. In these cases, f acts on the one-dimensional subspace spanned by $\vec{v}$ as either the identity operator or $F_{\vec{v}}$. What fact have we proved here?"

"Oh, oh! I know, Princess! You've proved that every flip operator can have a vector-flip factored out of it!"

"Why is that important?"

"Because that observation can be applied recursively on f! Every flip f can be written $f = F_{\vec{p}}\,f_{-1}$, where $\vec{v}_1$ is some vector on which f flips and f_{-1} is a flip that acts on the subspace perpendicular to $\vec{v}$. So you can write *any* flip as a product of perpendicular vector flips!"

"Right! Of course that only applies to *countable dimensional* inner product spaces."

Twilight nodded eagerly. The proof had really made her excited because she had actually seen where it was going before Luna had finished. She *was* getting smarter! The technique would never have occurred to her just a few days ago.

“And since each one dimensional flip means either multiplication by 1 or -1,” continued Luna, “You can just think of a flip in terms of two perpendicular subspaces. Those which are multiplied by 1 and those which are multiplied by -1.”

Twilight nodded.

“Now let’s think about one-dimensional flips that are *not* orthogonal. We shall first flip about $\vec{p}$ and then about $\vec{q}$.” Luna wrote on the board.

$$\begin{gathered}(1-2\,\vec{q}\otimes\vec{q})\,(1-2\,\vec{p}\otimes\vec{p})\\ 1-2\,(\vec{p}\otimes\vec{p}+\vec{q}\otimes\vec{q})+4\,(\vec{q}\otimes\vec{q})\,(\vec{p}\otimes\vec{p})\\ 1-2\,(\vec{p}\otimes\vec{p}+\vec{q}\otimes\vec{q})+4\,(\vec{q}\cdot\vec{p})\,(\vec{q}\otimes\vec{p})\end{gathered} \tag{4.10}$$

“Notice what happens when you multiply two outer products together. Verify that step using abstract index notation!” Luna said curtly.

“Very well, Princess.

$$\left(\vec{a}\otimes\vec{b}\right)\left(\vec{c}\otimes\vec{d}\right)=a_i\,b_m\,g_{mj}\,c_j\,d_n\,g_{nk}=\left(b_m\,g_{mj}\,c_j\right)(a_i\,d_n\,g_{nk})=(b\cdot c)\left(\vec{c}\otimes\vec{d}\right) \tag{4.11}$$

“No problem! Ehehehe!” Twilight laughed nervously.

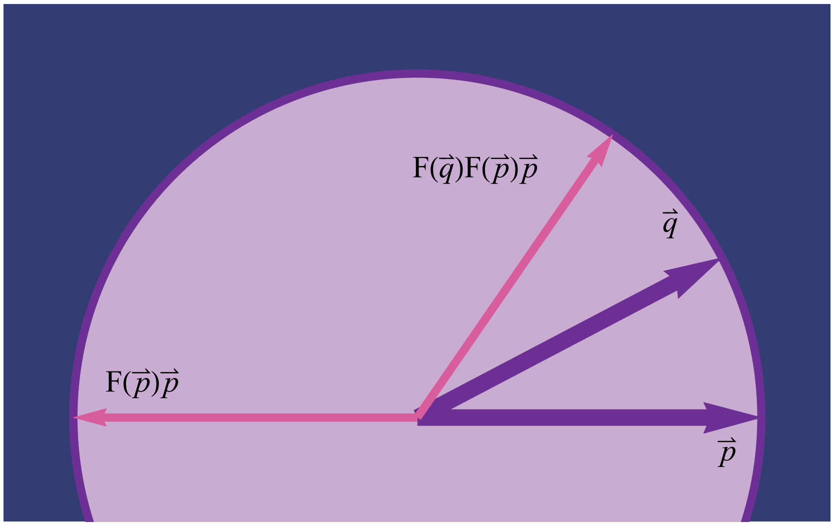
“Now clearly, if the two flip vectors were orthogonal, then the second term in formula 4.9 would be zero. This is an important sort of object, so let us decide what it is.”

“Yes, let’s! I want to know what it is, Princess!”

“We shall! Since only two vectors define the operation, we should only need to think about the plane defined by the two vectors and ignore any perpendicular vectors, correct?”

Twilight nodded. “It’s easy to see that if you multiply by a vector that is perpendicular to both $\vec{p}$ and $\vec{q}$, then the vector will not be altered.

“So we only need to think about two dimensions to understand what the sequential flip does. This diagram shows the result of flipping $\vec{p}$ about $F_{\vec{p}}$. The purple vectors are $\vec{p}$ and $\vec{q}$. The pink vectors show the result of performing the two flips on $\vec{p}$.



"It's a rotation like before, only now we can make any angle!" observed Twilight.

Definition 4.6 : *Planar Rotation*

"Quite so. Two nonorthogonal vector flips produce a rotation in the plane spanned by the two flips. I will define a *planar* rotation as a product of two vector flips. It must be an orthogonal transformation because it is a product of orthogonal transformations, but this is also easy to derive."

On queue, Twilight was already writing on the wall.

Proposition 4.3

$$F_{\vec{q}} F_{\vec{p}} \left(F_{\vec{q}} F_{\vec{p}}\right)^T = F_{\vec{q}} F_{\vec{p}} F_{\vec{p}} F_{\vec{q}} = F_{\vec{q}} F_{\vec{q}} = 1 \tag{4.12}$$

"A seemingly more difficult problem is what you get when you multiply two rotations. Consider two rotations $R_{\vec{v}\vec{w}}$ and $R_{\vec{p}\vec{q}}$. Now clearly if the plane spanned by $\vec{v}$ and $\vec{w}$ is perpendicular to the plane spanned by $\vec{p}$ and $\vec{q}$ (by which I mean every vector in the one is perpendicular to every vector in the other), then clearly $R_{\vec{v}\vec{w}}$ and $R_{\vec{p}\vec{q}}$ commute with one another, so there is no way to simplify that case further."

"Wait a minute," said Twilight. "How is it possible for every vector in one plane to be perpendicular to every vector in another plane? I can't imagine how that's possible."

"Sometimes it is better not to try to imagine things. That can become unnecessarily confusing."

“But they are both subspaces, right? So they would both have to include the zero vector, right? But they could not have any other vectors in common.”

“Yes, but that is alright because the zero vector is perpendicular to itself. But no, they can have no other vectors in common, since any other vector would not be perpendicular to itself.”

“But my question is, how is that possible? I can’t imagine two planes that intersect on only one point!”

“I see what you are asking, Twilight Sparkle! Two planes must intersect on a line, right?”

“Yes.”

“Wrong! That is only in three dimensions.”

“But I thought today we were doing geometry. How can there be a plane in four dimensions?”

“Just as easily as in three dimensions. Maybe even easier! In four or more dimensions, planes can intersect at a point.”

“I just don’t see how that’s possible.”

“You are trying to visualize it. Well you cannot. Sadly, your brain is not built for that, Twilight Sparkle! That is just why you must not take your intuitition *too* seriously. This is a case where it is wrong.

Now, what if $R_{\vec{v}\vec{w}}$ and $R_{\vec{p}\vec{q}}$ take place in the *same* plane? A rotation in two give dimensions is defined by an angle, and two rotations in the same plane should result in a rotation which is the sum of their angles. That should be clear if you imagine it, but do you have an idea how you would prove it?”

“True, but let us prove it more formally now.”

“Ummmm, ok. Oh, oh! I know the answer! If all for vectors are in the same plane, I can write $R_{\vec{p}\vec{q}} = R_{a\,\vec{v}+b\,\vec{w},\,c\,\vec{v}+d\,\vec{w}}$. Then, I can do something like

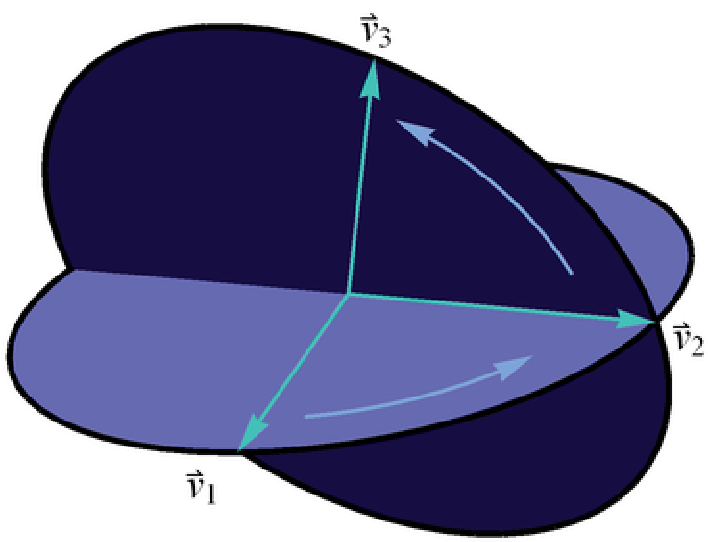
$$
\begin{aligned}
R_{\vec{v}\vec{w}}\,R_{\vec{p}\vec{q}}\,R_{\vec{v}\vec{w}}^{-1} &= F_{\vec{w}}\,F_{\vec{v}}\,F_{\vec{q}}\,F_{\vec{p}}\,F_{\vec{v}}\,F_{\vec{w}} \\
&= F_{\vec{w}}\,F_{\vec{v}}\,(c\,F_{\vec{v}} + a\,F_{\vec{w}})\,(a\,F_{\vec{v}} + b\,F_{\vec{w}})\,F_{\vec{v}}\,F_{\vec{w}} \qquad .\text{”}
\end{aligned}
\tag{4.13}
$$

“You have got the right idea,” said Luan. “You may fill the rest in later if you want. The important bit is this: two rotations in a plane commute and a planar rotation is invariant under planar rotations in its own plane. This is exactly what you learned earlier when you were flipping that picture.

Finally we are ready for the case in which the two rotations are neither perpendicular nor parallel! Here’s the trick. If the two planes span a 3–dimensional space, they must have a single 1–dimensional subspace in common, correct?”

“Yes,” said Twilight.

“Now,” continued Luna, “Because you can rotate a planar rotation in its own plane without changing the operator at all, or in other words, because a planar rotation depends on a plane and an angle, but not on any specific vector, we may now consider that $R_{\vec{v}\vec{w}}$ and $R_{\vec{p}\vec{q}}$ have been rotated such that $\vec{w} = \vec{p}$ is a vector in that shared subspace. That is what I have done in this diagram.



You see? First rotate from $\vec{v}$ to $\vec{p}$ and then from $\vec{p}$ to $\vec{q}$. Now find out what happens.”

Twilight wrote on the board.

Proposition 4.4

$$F_{\vec{q}} F_{\vec{p}} F_{\vec{p}} F_{\vec{v}} = F_{\vec{q}} F_{\vec{v}} \tag{4.14}$$

"Why," she said, "the result is another planar rotation!"

"That is right! But remember that the angles from $\vec{v}$ to $\vec{p}$ and $\vec{p}$ to $\vec{q}$ are only half the angles of the rotations. It is a different way to think about rotations than you might be used to.

We know now, though, that products of vector flips are always either general flips, products of planar rotations, or products of planar rotations and a flip. We are not yet in a position to prove this, but this also exhausts all the orthogonal transformations."

"That formula 4.9 is a little bit inconvenient," said Twilight. "I mean since the angle of the rotation is twice the angle of the two vectors. Isn't there a formula for a rotation in terms of two vectors that rotates one vector directly into another."

Definition 4.7 : *Symmetric operator*, *asymmetric operator*

"Yes," said Luna. "We suppose we could! That might even be a good lesson. Well the first idea to introduce is that every operator can be split into a *symmetric* and *asymmetric* parts. This is very easy.

$$B = B_S + B_A = \frac{1}{2}\left(B + B^T\right) + \frac{1}{2}\left(B - B^T\right) \tag{4.15}$$

The first part is the symmetric part and the second is the asymmetric part. You can see this because if you take the trace of the symmetric part, it comes out the same, and if you take the trace of the asymmetric part, it comes out opposite.

$$\begin{aligned} \left(\frac{1}{2}\left(B + B^T\right)\right)^T &= \frac{1}{2}\left(B^T + B\right) = B_S \\ \left(\frac{1}{2}\left(B - B^T\right)\right)^T &= \frac{1}{2}\left(B^T - B\right) = -B_A \end{aligned} \tag{4.16}$$

Symmetric and asymmetric operators have the interesting property that each retains its symmetry or asymmetry under orthogonal transformations. You can see this pretty simply with the orthogonal transformation O.

$$
\begin{aligned}
\left(O\, B_S\, O^T\right)^T &= O^{T^T}\, B_S^T\, O^T = O\, B_S\, O^T \\
\left(O\, B_A\, O^T\right)^T &= O^{T^T}\, B_A^T\, O^T = -O\, B_S\, O^T
\end{aligned}
\tag{4.17}
$$

You see? The symmetric part stays symmetric and the asymmetric part stays asymmetric!"

Twilight nodded.

"If you were a mathematician, you would say that this shows symmetric and asymmetric operators each form a representation of the group of orthogonal matrices."

"Ok..."

"Some day you'll think that's terribly profound. Now I want you to separate equation 4.9 into symmetric and asymmetric parts."

Twilight wrote on the board.

$$
1 - 2\left((\vec{p}\otimes\vec{p} + \vec{q}\otimes\vec{q}) - (\vec{q}\cdot\vec{p})\,(\vec{q}\otimes\vec{p} + \vec{p}\otimes\vec{q})\right) + 2\,(\vec{q}\cdot\vec{p})\,(\vec{q}\otimes\vec{p} - \vec{p}\otimes\vec{q}) \tag{4.18}
$$

"The first two terms are the symmetric part, and the last is the asymmetric part," she said.

"Let us say this represents a rotation about the angle θ. And let us define a new vector $\vec{r}$ which differs from $\vec{p}$ by the angle θ. The problem is now to somehow rewrite $R_{\vec{p}\vec{q}}$ expression in terms of $\vec{p}$ and $\vec{r}$.

Look at the asymmetric term first. We know that $\vec{p}\cdot\vec{q}$ is equal to $\cos(\theta/2)$. Now look at the operator $\vec{q}\otimes\vec{p} - \vec{p}\otimes\vec{q}$. This is actually a great opportunity to think of operators as vectors. What is the length of that operator, conceived as a vector?"

Twilight wrote on the wall.

$$
\begin{aligned}
&\sqrt{\mathrm{Tr}((\vec{q}\otimes\vec{p} - \vec{p}\otimes\vec{q})\,(\vec{q}\otimes\vec{p} - \vec{p}\otimes\vec{q}))} \\
&= \sqrt{\mathrm{Tr}(2\,(\vec{q}\cdot\vec{p})\,\vec{q}\otimes\vec{p} - (\vec{q}\cdot\vec{q})\,\vec{p}\otimes\vec{p} - (\vec{p}\cdot\vec{p})\,\vec{q}\otimes\vec{q}}
\end{aligned}
$$

"The trace of a outer product of vectors is their inner product," she said. "And there will be a factor of 2 in there which I can take out of the square root. So therefore,

$$\sqrt{2}\ \sqrt{(\vec{p}\cdot\vec{q})^2-1} = \sqrt{2}\ \sqrt{\cos(\theta/2)^2-1} = \sqrt{2}\ \sin(\theta/2) \qquad .\text{”}$$

“So!” said Luna, “just ignore the factor of $\sqrt{2}$ for now. You can think of the asymmetric term as being like $2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$. Now remember the trig identity

$$\sin(\theta) = 2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \qquad .$$

If $\sin(\theta/2)$ is a lot like $\vec{q}\otimes\vec{p}-\vec{p}\otimes\vec{q}$, then we should expect that $\sin(\theta)$ should be just like $\vec{p}\otimes\vec{r}-\vec{r}\otimes\vec{p}$. That means we can replace the term $2\,(\vec{q}\cdot\vec{p})\,(\vec{q}\otimes\vec{p}-\vec{p}\otimes\vec{q})$ with $\vec{r}\otimes\vec{p}-\vec{p}\otimes\vec{r}$.

And what about the other term? Let me just suggest something to get you thinking a little more. A rotation matrix in two dimensions looks like this.

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

You can verify that that works when you get home. So we’re trying to find something that looks kind of like that, only it can work along some arbitrary two–dimensional subspace of some higher–dimensional space. Make sense?”

“I guess so.”

“Now we already found the a part that looks like the sines in that matrix. We need to find something that works like the cosines, right?”

“Ok...”

“So let us think of the second term in expression **RotationFormulaSplit** as being like $1-\cos(\theta)$. That way, when it adds to the identity at left side of **RotationFormulaSplit**, it might works out to be a nice $\cos(\theta)$. Now recall the trig identity

$$1-\cos(\theta) = 2\sin\left(\frac{\theta}{2}\right)^2 \qquad .$$

So the *second* term—the really complicated one—we can think of as being both like $1-\cos(\theta)$ and *also* like $2\sin\left(\frac{\theta}{2}\right)^2$. But *that* means we could *also* think a similar expression $(\vec{p}\otimes\vec{p}+\vec{r}\otimes\vec{r})-(\vec{r}\cdot\vec{p})(\vec{r}\otimes\vec{p}+\vec{p}\otimes\vec{r})$ as being something like $\sin(\theta)^2$. What could we do to it to turn it into something more like $1-\cos(\theta)$, except written in terms of $\vec{p}$ and $\vec{r}$?

Twilight said, "Err… this isn't making a lot of sense. I mean how—"

"Twilight Sparkle, you must have faith in your teacher! Now follow my lead," Luna said sweetly.

Twilight creased her brow and sighed a little. "Alright, Princess. Well if that term is *like* $\sin(\theta)^2$ then I could multiply by $(1-\vec{r}\cdot\vec{p})\Big/\left(1-(\vec{r}\cdot\vec{p})^2\right)$ because the denominator is equal to $1-\cos(\theta)$ and the numerator is equal to $1-\cos(\theta)^2=\sin(\theta)^2$."

"That's right! Exactly what I was thinking. So now let us write the new expression

$$\begin{gathered}1-((\vec{p}\otimes\vec{p}+\vec{r}\otimes\vec{r})-\vec{r}\cdot\vec{p}\,(\vec{r}\otimes\vec{p}+\vec{p}\otimes\vec{r}))\,(1-\vec{r}\cdot\vec{p})\Big/\left(1-(\vec{r}\cdot\vec{p})^2\right)+\\(\vec{r}\otimes\vec{p}-\vec{p}\otimes\vec{r})\end{gathered}\quad, \tag{4.19}$$

and there we have it!"

"Really?" said Twilight. "I don't think you've proved that *at all*!"

"You can prove it is right by confirming that it works the same as $R_{\overrightarrow{pq}}$, but that will not be a terribly enlightening exercise."

Twilight nodded. "Er, ok. It was a very strange process we used to arrive at ind and it's actually not a very nice–looking formula in comparison to the other one. Maybe it makes more sense to think of rotations in terms of *half* the rotation angle, since they multiply so nicely that way."

"Very good! But deriving the new formula was a good exercise because you learned a valuable lesson. I did not actually *derive* that expression. I simply made a bunch of completely extremely suspicious leaps of fancy and constructed a formula based on that. Then I verified that it was the correct formula."

"What exactly is the lesson?"

"That you can *do* that."

"Ohhhh."

"Does this equation look scary to you?"

"A little, Princess."

"But but not terrifying."

"No."

"Why not?"

"Because—I understand what the equation means and what it's for."

"Right. The fact that it looks complicated is incidental for you now. It's a tool that you know how to use, and I gave you some analogies to help make it seem familiar. A bit of mathematical mythology."

"Yes, you did, Princess. Thank you!"

"That's enough for this morning I think." said Luna after a slightly awkward pause. "Therefore, I will see you tomorrow." Her horn began to glow as she summoned up a black hole.

"Wait!" said Twilight.

"What is it?" asked Luna

"Well I was wondering if we could talk about something else for a moment."

"Why, what else would we ever wish to talk about?"

"I was just wondering how things were going with you."

"Going?" Luna seemed confused for a moment. "Well, I just started reading this book on algebraic geometry…"

"No!" yelped Twilight before catching herself and calming down. "No… I mean, how are you feeling? Did you have a good night and morning?"

Luna looked confused for a minute. "Good morning… good night… Hmm. Yes. Yes, I think I did. Well! Thank you, Twilight Sparkle! I'm glad we had this talk!"

“No! That wasn’t—” but Luna had already disappeared in a flash of gravitons.

“… a real talk.” Twilight stompped her hoof. “She is just *impossible*!”