Theoretical Physics is Magic

Smeapancol

Unit 1: Space

Lesson 1- Algebra

Day 1- Abstraction

In a neglected corner of the Canterlot Archives, in an alcove at the end of a long hall lined with old books, Twilight Sparkle sat reading by the light of her horn, muttering to herself at the open book in front of her.

“Complex Hilbert space? Holonomy groups? What? Infinitely differentiable Riemannian manifolds? Vector… bundles—*what*?? Analytic continuation?! This doesn’t make any sense!”

Finally she slammed the book shut in frustration and tossed it on the floor.

“This… is… *impossible*! I don’t understand. I just don’t understand!” she cried, becoming more and more agitated. “This is nothing like astronomy! How can anyone understand it!? I don’t understand how I can not understand!”

Twilight let her head droop over her forelegs. She sighed and felt tears coming on. But then her ears perked at some commotion from down the hall and she hoped no one overheard her almost go insane again. She heard a familiar bellow.

“MAKE WAY FOR THE PRINCESS OF THE NIGHT! WE ARE LOOKING FOR A BOOK TO READ THIS EVENING!”

Twilight buried her face in her hoof. After all this time and Princess Luna still didn’t understand how to behave in a library! At least there weren’t many ponies around that late.

“GUARDS! STAND UP STRAIGHTER! THAT IS ACCEPTIBLE. NOW THEN, THIS ONE HAS TOO MANY PAGES! THIS ONE IS TOO LONG!”

Twilight groaned and wiped her eyes as the voice came closer. She saw the Princess’s glow from around the corner and tried to hide behind another book. However, the page she opened contained a particularly nasty–looking integral. She gulped when she saw it and dropped the book in fright.

“TWILIGHT SPARKLE! HELLO! THAT LOOKS LIKE JUST THE PLACE FOR YOU!”

She tried to collect herself as the Princess approached, but she could still feel herself shaking with frustration at herself.

“WHAT IS IT, TWILIGHT! ARE YOU FEELING…?”

Twilight waited for Luna to finish her sentence until she realized that Luna had no clue what she was feeling. “Princess… the voice, remember?”

“Oh yes, I keep forgetting. That feels like my normal way of talking.”

“What are you even doing here? Aren’t you guardian of the night?”

“Have you been reading here all night? It is dawn now. I just came off duty.”

Twilight groaned. “I guess I have. And nothing to show for it either.”

Luna noticed the book on the floor and lifted it with her telekinesis.” Oh! You are reading *Quantum Mechanical Theory*, the classic exposition by Atomic Force and Light Speed, two of the greatest physicists of our generation. I did not know you did physics too. Is there anything you do not know, Twilight Sparkle?”

“That’s just it, Princess! I *don’t* know it. No matter how hard I try, I just can’t follow it!”

“Oh. Well do not feel bad about *that*. Most ponies could never understand physics no matter how hard they tried!”

“Because I didn’t think there was anything I couldn’t understand! That’s never happened to me with anything before.”

Luna snapped her head to the side, making shadows dance across the room with her dimly glowing mane. “I UNDERSTAND NOW! YOU ARE ASHAMED OF BEING INEPT IN PHYSICS!”

Twilight buried her head under her hooves, certain that her secret was out now. “Princess! Please don’t use the Royal Canterlot voice. Everyone will hear you.”

“How silly of me. I keep forgetting,” Luna said with an elegant laugh.

“I’ve been trying to read this book since before I moved to Ponyville. I don’t understand it! Now I have trouble even looking at the book because the equations give me so much anxiety! I don’t know how to deal with something that I have so much trouble understanding!”

“There there, poor dear! It *is* a difficult subject. I had trouble with it myself!”

TWILIGHT: You mean, *you* know physics?

“Are you surprised? Goddesses have to know these things! Plus, there wasn’t a lot to do during my time on the moon, and studying physics was one of the things I could do. When I wasn’t plotting my revenge, of course.”

“Er, yes—of course.” Twilight tried giving Luna her most adorable puppy eyes. “Do you think you might be able to help me get started on this?”

Luna appeared to be in deep thought for a moment and then stomped her forehoof with “VERY WELL, TWILIGHT! IN RETURN FOR SHOWING ME THE MAGIC OF FRIENDSHIP, I DECREE THAT YOU SHALL APPEAR HERE EVERY EVENING TO STUDY PHYSICS.”

“Every night? I don’t think I have time for—”

“You are not turning down MY GENEROSITY are you?”

“Well no but—”

“It’s settled then! Your lessons begin… NOW!”

With a flash of Luna’s dark maroon aura, a large blackboard appeared in the hall, blocking the way out from floor to ceiling.

“Tell me, disciple, how much calculus have you done?”

“I’ve done a bit of calculus. That’s not so hard because it’s just memorization.”

“The reality is more subtle than that. Doing calculus is not just about memorizing rules but also pattern recognition, and that is the hard part. For example, take something like the integral

$$\int dx \, \frac{1}{\sqrt{x}\,(1+x)} \quad . \tag{1.1}$$

One could try doing this with integration by parts, but that will not get far. The trick is to notice that it can be can writen it like this.

$$\int dx \, \frac{1}{\sqrt{x}\left(1+\left(\sqrt{x}\right)^2\right)} = 2\int dx \, \frac{1}{1+\left(\sqrt{x}\right)^2} \frac{d}{dx}\sqrt{x} = 2\int du \, \frac{1}{1+u^2} \tag{1.2}$$

And now you can see that it is really just a chain rule problem, and you recognize the derivative of $2\arctan\left(\sqrt{x}\right)$.

In this problem it is not at all obvious which rule will actually work to simplify the expression. That is what makes *all* mathematics challenging!”

Luna finished writing out the integral and turned back to face Twilight, whereupon she gaped in shock. Twilight was on the ground, lying stiff and twitching”

“Oh dear Twilight Sparkle! WHAT’S WRONG?”

“I—integral,” whimpered Twilight.

Luna quickly erased the integral. “I had no idea your problem was so severe! There, it’s gone! See? No integral.”

“All gone?” Twilight murmured, still on the ground.

“All gone!”

Twilight shook her head and stood up again. “Sorry Princess… You see? It makes me too nervous!”

“We had better do something easier. How about addition and multiplication?”

Twilight laughed nervously. “I can do figures in my head! I think I could do something a *little* more advanced.”

“This is not that kind of arithmetic. We shall think about addition abstractly, without knowing what we are adding. We shall think about it in terms of the *properties* of addition, not in terms of the result. But do not think of addition as a procedure. Think of it more as a structure.

Now,” said Luna as she began to write on the board again, “addition has four properties: associativity, which allows us to treat a sum of any number of symbols as a single operation; commutativity, which, together with associativity, allows us to treat any finite sum as a homogeneous pile, identity, which says that there is an element 0 which can be added to anything without changing the result; and inverses, which says that everything has an inverse that can be added to it to produce 0.

Definition 1.1 : *Addition*

$$\text{Associativity} — (a+b)+c = a+(b+c) = a+b+c$$

$$\text{Commutivity} — a+b = b+a$$

$$\text{Identity} — a+0 = a$$

$$\text{Inverse} — a-a = 0$$

Let us warm up by trying to prove that the identity is unique. Let’s say that 0 and $\overline{0}$ are both identity elements. In this case the line over the 0 is just something to distinguish it from the other 0. Use the properties of addition to prove that they are equal. Do you think you can do that for me?”

Twilight grinned. “I think I can handle this one! Well the only thing you can do to start off is add 0 to anything. And then I can just use the reverse of the identity rule to remove the $\overline{0}$.”

Proposition 1.1

$$\overline{0} = \overline{0} + 0 = 0$$

“Good! But you made a slight omission. You will notice that the axioms only allow adding 0 on the right. So you ought to have used the commutativity rule to swap the 0 and $\overline{0}$ before you could remove the $\overline{0}$.”

Proposition 1.2

$$\overline{0} = \overline{0} + 0 = 0 + \overline{0} = 0$$

“Hmm. But Princess, wouldn’t there have to be a similar rule for $=$? I mean I applied the identity rule twice, but the second time I applied it backwards. So by your logic, wouldn’t we have to have both $a + 0 = a$ and $a = a + 0$ as axioms?”

Luna was momentarily flustered. “Well! My answer is that we are treating the $+$ relation as an abstract operation that is defined only by its properties. Whereas the $=$ relation is actually meaningful, and it stands for equality, which I am assuming you know is by nature reflexive, symmetric, and transitive.”

“I see. That’s a very fine philosophical distinction!”

“ONWARD! I mean, onward. Now, like everything, 0 has an inverse. Prove that -0 and 0 are equal.”

“Once again there aren’t many choices available for what to do.”

Proposition 1.3

$$-0 = -0 + 0 = 0 - 0 = 0$$

“First identity, then commutativity, and then inverses. Good.

Finally, prove that the inverse of each element is unique. Start by assuming $-a$ and $-\overline{a}$ are both inverses of a.”

“I think I’m getting the hang of this now.”

Proposition 1.4

$$-\overline{a} = -\overline{a} + 0 = -\overline{a} + a - a = -a$$

“The last thing to prove is that $-(-a) = a$.”

“Ok.”

Proposition 1.5

$$-(-a) = -(-a) + 0 = -(-a) - a + a = 0 + a = a$$

“Notice you had to use commutativity again for that one. Now you know everything there is to know about addition!

Now on to multiplication. Multiplication always obeys the rules of associativity and distributivity, but there is not necessarily any assumption about identities and inverses.

Definition 1.2 : *Multiplication*

$$\text{Associativity} — (a\,b)\,c = a\,(b\,c) = a\,b\,c$$

$$\text{Left distributivity} — a\,(b + c) = a\,b + a\,c$$

$$\text{Right distributivity} — (a + b)\,c = a\,c + b\,c$$

Also, since we do not have commutativity, a multiplication operation needs to be understood as an ordered list rather than a lump, as with addition. This means that multiplication can sometimes only work one way. If $a\,b$ is defined, this does not mean that $b\,a$ is always defined.

If there is no identity, there cannot be inverses because the concept of an inverse depends on that of an identity. Now sometimes there *is* an identity defined for multiplication, but there is not always both a right and a left identity."

Definition 1.3 : *Multiplicative identity*

$$\text{Left Identity} — 1\,a = a$$

$$\text{Right Identity} — a\,1 = a$$

Twilight nodded. "Before, we had to use the fact that the identity is both a right and a left identity to prove that it's unique."

"Indeed. When inverses are not assumed, we will often be preoccupied with the question of characterizing inverses and inverses and determining when they exist.

Now, something important happens when you multiply by zero. Show me what that is."

"There is more going on now that we have both multiplication and addition, but I think I got it.

Proposition 1.6

$$0\,a = (b - b)\,a = b\,a - b\,a = 0$$

And the proof that $0\,a = 0$ would be the same."

"This works as long as you can always find some b to multiply by any a. So it always works, for example, when there is an identity because you can just set $b = 1$.

Next think about what happens when you multiply by -1."

"These all seem like pretty obvious properties of numbers, and I mean I already know that will produce the additive inverse, so why don't we just skip that one?"

"YOU WILL SOLVE EVERY PROBLEM I GIVE YOU!"

"Eep! Ok, ok! P—princess!"

Proposition 1.7

$$(-1)\,a = 0 + (-1)\,a = -a + a + (-1)\,a =$$
$$-a + 1\,a + (-1)\,a = -a + (1-1)\,a = -a + 0\,a = -a + 0 = -a$$

"Ahem. Good. For that you had to use the property that everything times 0 is 0. And from this we also know that $(-1)\,(-1) = -(-1) = 1$.

Now we can talk about multiplication with inverses. Try to stay calm because I'm going to use some new symbols!"

Definition 1.4 : *Multiplicative inverses*

$$\forall\, x \neq 0\, \exists\, x^{-1} \ni\ x\,x^{-1} = x^{-1}\,x = 1$$

This says, if x is nonzero, there is an inverse x^{-1} such that the product is 1. I am just putting the right inverse and left inverse together in one axiom because I do not know of any cases in which we will have only a left inverse or only a right inverse."

Proposition 1.8

Twilight breathed deeply and partly blocked the view of the blackboard with her hooves, trying to only see part of it at a time. "Got it!" She thought for a moment. "And since inverses commute, the same proof that $-(-a) = a$ would work to prove that $\left(a^{-1}\right)^{-1} = a$."

"Right! Do you see why 0 cannot have an inverse?"

Twilight smirked a little. “I think it *can* have an inverse!

$$1 = 0\,0^{-1} = 0$$

But only if $0 = 1$.”

“Which implies?”

“Err… it implies…”

$$a = 1\,a = 0\,a = 0$$

“Yes, everything is zero.

The next thing we need to talk about is the concept of closure. That means that there is a definite set of elements, and an operation is defined for every member of the set. And the operation always stays within the set. In other words, for a set S, we might define a function $f : S \times S \to S$, which satisfies the properties of addition or multiplication.

Now I shall mention some objects that are defined by various combinations of the properties we discussed today, but that is just for reference. We will not do anything with them today.

Definition 1.5 : *Ring*

A set that has both closed addition and closed multiplication with left and right identity is called a *ring*.

By the way, quick aside. If a set is closed under addition or multiplication for binary sums or products (which is how I defined closure), does this mean it is closed under infinite sums or products?”

“Er, yes?”

“No. Infinite sums or products of a set whose elements are defined to have certain properties need not satisfy those properties. The rational numbers, for example, are closed under addition, but infinite sums of rational numbers can be real numbers.”

“I hope we won’t have to do too many infinite sums…”

“Not for a while. But you will learn to love them!

Definition 1.6 : *Group*

A set which has only closed multiplication with inverses defined on it is a *group*. Since a group does not have addition, there is no 0 defined in it, so everything has an inverse.

“But didn’t the axioms you told me earlier say that there’s always a 0?”

“You can think of 0 as existing somewhere out in the void, but just not within the set defining the group. Eventually we will be multiplying by zero though.

Definition 1.7 : *Field*

A *field* is closed under multiplication and addition and normally has the additional axiom that multiplication is commutative. If you skip the commutativity axiom, it is a *noncommutative field*.

I should mention that mathematicians and physicists have a different meaning for the world *field*. This is the mathematical definition. What physicists call a field, mathematicians would say is a kind of commutative module. So most of the time when we talk about fields we will be talking about something else.”

A nearby potted plant rustled and suddenly Pinkie Pie’s head burst out from beneath it, the plant sitting atop her head. “Like for example, all the fields mentioned in the table of contents for the rest of this whole story!”

Twilight yelped in shock. “What are you talking about?? And how’d you even get in here, Pinkie?”

Pinkie’s head slowly submerged again. “There is no explaining Pinkie Pie!”

“Some day, Pinkie! Science has the answers!” Twilight screamed at the pot.

“...Well that was odd,” said Luna. “Anyway... fields are very familiar objects because they have all the ordinary arithmetical operations defined on them: addition, subtraction, multiplication, and division. In the future, if I talk about numbers, I will usually be referring to the element of some field, so it will be assumed that addition, multiplication, and division are allowed. The rational numbers, the real numbers, and the complex numbers are popular examples. Of course physicists do not bother with rational numbers.”

Twilight chuckled. “Of course physicists only use real numbers. Those are the only numbers that exist in real life!"

Luna scowled. "Ha. Ha. Ha. FOOL! Just for that, I have another problem for you! I suppose you know that $i^2 = -1$, but now you must tell me what i^{-1} is."

Twilight gulped and turned to the board. "No problem, Princess."

Proposition 1.9

$$i^{-1} = i^{-1}\,1 = i^{-1}\,(-1)\,(-1) = i^{-1}\,i\,i\,(-1) = i\,(-1) = -i$$

"Quite so. From now on you do not need to do these tedious proofs with every step written out. Now you may simply say, 'since $(-i)\,i = -(-1) = 1$, then i and $-i$ are multiplicative inverses.'"

"Thank you, Princess. That's a relief!"

"Now there is always a natural sense in which things that can be added together can be multiplied by the integers. You write something like

$$1\,a = a$$
$$2\,a = a + a$$
$$3\,a = a + a + a$$

and so on.

Now extend this idea so that we have some set whose elements can be multiplied by the numbers *any* field, so we might not just have $2\,a$ and $3\,a$, but also $\frac{1}{2}\,a$ and And maybe even $(i + 7)\,a$, depending on what the field was."

Twilight pressed her hooves to her temples. "OK, I'm extending the idea in my mind."

"That idea is basically a vector space."

"Actually, I think I've heard of vectors before. Aren't they things with both a magnitude and direction?"

"NO!" Luna yelled. "Those are nothing but… lies! Propaganda promulgated by… by rank amateurs!"

Twilight was taken aback by Luna's reaction. "I'm sorry I just read that—"

"BURN THE BOOK YOU READ IT IN THEN!"

Twilight sat awkwardly for a moment as she waited for Luna's anger to calm down.

"I'm sorry Princess. Is everything alright?"

"I am sorry, Twilight Sparkle, it is just that you have uncovered a horrible ambiguity in our terminology. I did not wish to have to burden you with this, but what most people call a vector is actually a representation of the rotation group, which also happens to be a vector as well, but there are lots of other vectors that aren't characterized by having a magnitude and direction, even some with a completely different geometrical interpretation."

"Er, I see..."

"And there are lots of other vectors that are not geometrical in the slightest."

"I'll keep that in mind!"

Definition 1.8 : *Vector space*

"Right. Now a *vector space* V over a field F is a set whose elements are called *vectors*, understandably enough. Vectors have a addition defined on them and a kind of multiplication. However, you cannot multiply a vector with a vector. You can only multiply a vector by an element of its field. This is called *scalar* multiplication. A vector space is closed under addition and scalar multiplication. That is what a vector space is.

You already know about multiplication and addition, so you already know the most basic properties of vector spaces."

Twilight recited what she had learnd. "There's a unique identity element 0 that gives 0 when multiplied by any scalar. The field has unique and distinct elements 0, 1, and -1, and it's got additive and multiplicative inverses. So, for scalar multiplication, there's a unique left multiplicative identity 1, but not a right multiplicative identity. And of course vectors don't have multiplicative inverses since they can't be multiplied."

"Indeed!"

"Excuse me, but isn't the symbol 0 ambiguous here? I mean there's both an additive identity in the vector space and in the field, and they're logically distinct from one another. If v is a vector and s is a scalar, you could write

$$\begin{aligned} 0\,v &= 0 \\ s\,0 &= 0 \end{aligned} \quad .$$

In the first equation, the first zero is the zero scalar and the second is the zero vector, and in the second it's not even clear *which* zero that could be!"

"There will be no ambiguity because all zeros behave the same, so you never actually need to worry about it. On the other hand, you could think of it another way. There is only *one* 0 anywhere, and it has the property of being an additive identity for everything and can be multiplied by everything to give itself. When it is used with vectors, it acts like a vector, and when it is used with scalars, it acts like a scalar."

"But which is the *correct* way?"

"Oh, Twilight! Either way is just fine."

Twilight wondered how Luna could throw a tantrum about terminology but have no opinion about a seemingly much more realistic philosophical issue.

Definition 1.9 : *Module*

"By the way," continued Luna, "you can also define something like a vector space except over a ring rather than a field, and that is called a *module*. We'll won't need to go into *that* theory, but I just mentioned it for completeness. Now then!" Luna said with a strong wave of her forefoof, "I'll see you tomorrow for your next lesson."

"We're done? But... we didn't even do any physics!"

"You're not ready for physics yet. Perhaps in a few days we may be able to do some of the most elementary physics conceivable!"

"Wait! I have to thank you, Princess! You *did* make me feel more confident because you made me practice with easy things and you gave me advice on how to think about think about them. I haven't read a book that did that before."

"Some day, you will understand. And Huzzah! I have a disciple now too!" In a negative flash, Princess Luna disappeared into a black hole and took her blackboards with her.

Twilight was pleased at how much she had done without having a panic attack, but then she began to wonder what she had gotten herself into.